

A study of Quantum Correlations in Open Quantum Systems

Indranil Chakrabarty,^{1,*} Subhashish Banerjee,^{2,†} and Nana Siddharth^{2,‡}

¹*Institute of Physics, Sainik School Post, Bhubaneswar-751005, Orissa, India*

²*Chennai Mathematical Institute, Padur PO, Siruseri- 603103, India*

In this work, we study quantum correlations in mixed states. The states studied are modeled by a two-qubit system interacting with its environment via a quantum non demolition (purely dephasing) as well as dissipative type of interaction. The entanglement dynamics of this two qubit system is analyzed. We make a comparative study of various measures of quantum correlations, like Concurrence, Bell's inequality, Discord and Teleportation fidelity, on these states, generated by the above evolutions. We classify these evolved states on basis of various dynamical parameters like bath squeezing parameter r , inter-qubit spacing r_{12} , temperature T and time of system-bath evolution t . In this study, in addition we report the existence of entangled states which do not violate Bell's inequality, but can still be useful as a potential resource for teleportation. Moreover we study the dynamics of quantum as well as classical correlation in presence of dissipative coherence.

PACS numbers: 03.65.Yz, 03.65.Ud, 03.67.Mn

INTRODUCTION

Entanglement lies at the heart of quantum mechanics. In the last few years a lot of research has been done in the field of detection and quantification of entanglement. The original method for detection of entanglement was Bell's Inequality [1, 2]. Bell's inequality is derived from assumptions of a general, local hidden-variable theory (LHVT). This inequality is unique in the sense that all local hidden variable models satisfy it. In other words, the inequality derived by Bell has clearly demonstrated that some quantum mechanical predictions, in the ideal scenario, cannot be reproduced by any LHVT. Not only that, the violation of Bell's inequality is a strong signature for the inseparability of a quantum state. However, Bell's inequality is unable to detect all possible entangled states as there are examples of entangled states which does not violate it [3]. Thus one can conclude that violation of Bell's inequality implies entanglement, but the presence of entanglement does not necessarily imply violation of the inequality.

There is thus a need for a measure that can quantify the amount of entanglement present in the system. Out of many measures a very convenient one is concurrence [4]. Apparently, it may look that there is no difference in quantifying inseparability and quantumness. However, in general, it is not so. Even though it is quite well established that entanglement is essential for certain kinds of quantum-information tasks like teleportation and super-dense coding, the precise role of entanglement in quantum information processing still remains an open question. It is not clear, whether all information-processing tasks that can be done more efficiently with a quantum system than with a comparable classical system require entanglement as a resource. Indeed, there are instances where it is possible to do more with quantum rather than the classical, in the absence or near absence of entanglement [5–9]. This tells us that one cannot ascribe the mere absence of entanglement as a signature of classicality. As an example, we can talk about the DQC1 (deterministic quantum computation with one bit) model where the system has very little entan-

glement, and even that little vanishes asymptotically, yet it provides an exponential speedup [9, 10].

This raises an important question that if not entanglement what explains all the quantum advantages? Most likely, it has something to do with the structure and inherent non-locality of quantum mechanics. Interestingly, it is known that non-locality and entanglement are not equivalent features. Entanglement is the feature that having complete information about the subsystems does not reproduce the complete information of the whole system. Not surprisingly, this is not the only characterization of a quantum system. For instance, the collapse of one part of a subsystem after measurement of another is another feature which is unique to quantum systems and does not have any classical analogue. The quantity that tries to capture this unique feature is the quantum discord [11–13].

Quantum discord tries to quantify all types of quantum correlations including entanglement. It must be emphasized here that discord supplements the measures of entanglement that can be defined on the system of interest. It aims to capture all the non classical correlations present in a system, including entanglement. Other measures of quantum correlation, similar to discord, are the quantum dissonance [14], that aims to capture quantum correlations in separable states and measurement induced disturbance [15].

It may appear that analyzing Bell's inequality, concurrence, quantum discord for a two-qubit bipartite system is an exhaustive way of analyzing the inseparability along with quantumness of the system, but there is an applicational aspect to it. One of such application is teleportation.

In the early nineties a new aspect of quantum entanglement was discovered, viz. teleportation [16]. Teleportation is purely based on classical information and non-classical Einstein-Podolsky-Rosen (EPR) correlations [17]. The basic scheme of teleportation is to transfer an arbitrary quantum state from sender to receiver using a pair of particles in a singlet state shared by them. When two parties share a mixed state instead of a singlet, it is not possible to teleport an unknown quantum state with full fidelity. The natural question was to find out the optimal value of teleportation fidelity of

an unknown quantum state; a fidelity above which will ensure non-classical character of the state forming the quantum channel. It was shown that for a purely classical channel the optimum teleportation fidelity is $F = \frac{2}{3}$ [18–20].

In Ref. [21], the following question regarding quantum teleportation, Bell-CHSH (Clauser-Horne-Shimony-Holt) inequalities and inseparability was raised: What is the exact relation between Bell inequalities violation and teleportation? Bell inequalities are basically built upon the locality and reality assumption and have nothing to do with quantum mechanics. It is quite obvious that the state which violates Bell's inequality can be useful for teleportation. However, in another work, Werner [3] gave an example of an entangled state, which has the unique feature of not violating Bell's inequality for a certain range of classical probability of mixing. An interesting question to ask was whether states which do not violate Bell-CHSH inequalities are suitable for teleportation. An answer to this question was offered in the form of examples of mixed two spin- $\frac{1}{2}$ entangled states that do not violate Bell-CHSH inequalities but can still be useful for teleportation [19, 22, 23]. However any state that violates Bell-CHSH inequalities is always suitable for teleportation. In a recent work [24], these relations for two qubit mixed entangled state arising from the Buzek Hillery quantum cloning machine were probed.

An issue of central importance is the study of quantum correlations in mixed states. Open quantum systems [25] provide a natural setting for a systematic discussion of mixed states. A basic motivation of this work is to find out whether there are mixed, entangled states (having positive concurrence) which do not violate Bell's inequality but can still be useful for teleportation. We consider entangled states generated by open system models involving quantum non demolition (purely dephasing) as well as dissipative type of system-reservoir interaction. In addition, we make a comparative study of various measures of quantum correlations like concurrence, Bell's inequality, quantum discord and teleportation fidelity on mixed states, generated by the above evolutions. Interest in the relevance of open system ideas to quantum information has increased in recent times because of the impressive progress made on the experimental front in the manipulation of quantum states of matter towards quantum information processing and quantum communication. Myatt *et al.* [26] and Turchette *et al.* [27] have performed a series of experiments in which they simulated both pure dephasing as well as dissipative evolutions, by coupling the atom (their system- S) to engineered reservoirs, in which the coupling to, and the state of, the environment are controllable.

There has been considerable interest, in recent times, in the study of quantum correlations in open quantum systems. Quantum and classical correlations have been studied, in the context of quantum phase transitions [28]. An operational measure of quantum correlations, based on the dynamics of these correlations in open system evolutions, was proposed in [29], while in [30], a study of classical and quantum correlations was made, on a two-qubit system interacting with two

independent environments. A study of quantum discord and entanglement was made, on a system consisting of two coupled quantum dots in [31], and it was shown that discord can be more resistant to dissipation than entanglement. In [32], Markovian dynamics, and in [33], non-Markovian dynamics of quantum discord was made on a two qubit system, and it was seen that, in some conditions, quantum discord and entanglement can behave very differently. Interesting experimental investigations have also been made in this context. In [34], a theoretical and experimental study was made on the dynamics of classical and quantum correlations in an NMR quadrupolar system, while in [35], the dynamics of different kinds of bipartite correlations was investigated, experimentally, in an all-optical setup. In [36], an interesting experiment was presented, in which dissipation induces entanglement between two atomic objects, thereby paving the way for long-lived entanglement, in a steady state. In a recent work, it was shown that almost all quantum states have a non zero quantum discord [37]. In reference [38], a numerical comparison between discord and entanglement of formation was made. Some authors investigate the dynamical relations among entanglement, mixedness and non locality, quantified by concurrence C , purity P and maximum of Bell function B , respectively, in a system of two qubits in a common structured reservoir [39]. In another work, the dynamics of quantum and classical correlations in presence of non dissipative decoherence is studied [40].

In another important work [41] the authors a general characterization of quantum discord and entanglement in large families of states (Bell Diagonal states) has been presented. This also gives a kind of a general analysis of the structure of entanglement and discord in particular of their non analytic behaviour under decoherence. However, in this work we also provide a general analysis of the various measures of quantum correlations for the states that are modeled by a two-qubit system interacting with its environment via a quantum non demolition (purely dephasing) as well as dissipative type of interaction.

The basic motivation of the paper is three fold:

1) The primal objective is to classify the set of two qubit states modeled by a two qubit system interacting with its environment via a quantum non demolition as well as dissipative type of interaction. We categorize these density matrices into three broad divisions [14]:

- i) *Entangled states*: The density matrices having positive concurrence.
- ii) *Non classical separable states*: The states with a non zero discord and zero concurrence (absence of entanglement).
- iii) *Classical states*: The separable states with zero discord.

In this process of classification of the generated states our aim is to give a systematic overview of the threshold values of the dynamical parameters at which the transition from one class of states to other class is taking place.

2) Next, our motivation to study teleportation fidelity and Bell's inequality of the states, generated by this open system model, is to find out entangled states which do not violate

Bell's inequality but can still be useful for teleportation.

3) Last but not the least our objective is to see the interface of quantum and classical correlations in the presence of dissipative decoherence. In other words, it remains interesting to study the classical as well as quantum decoherence in presence as well as absence of entanglement.

The plan of the paper is as follows. We discuss various measures, both from a fundamental as well as an applicational perspective, of quantum correlations and try to bring out their interconnections. Two-qubit systems, used here in our study of quantum correlations in mixed states, are then discussed, briefly, from the perspective of a general division of open quantum systems into a purely dephasing (non demotion) or dissipative evolution. This is then applied to a study of quantum correlations, followed by our discussions and conclusions.

AN OVERVIEW OF MEASURES OF QUANTUM CORRELATION: BELL'S INEQUALITY, CONCURRENCE, DISCORD AND TELEPORTATION FIDELITY

In this section we try to analyze different measures of quantum correlation and make a comparative study between them. Among the measures studied are Bell's inequality, concurrence and quantum discord, from a theoretical perspective. In addition, we also discuss teleportation fidelity as a measure, from an applicational point of view.

Bell Inequalities

Bell's inequality was one of the first tools used to detect entanglement. Originally, Bell inequalities were introduced as an attempt to rule out local hidden variable (LHV) models.

Consider a bipartite system of two qubits where Alice and Bob share a particle, each supplied and initially prepared by another party, say, Charlie. Each of them are allowed to perform measurements on their respective particle. Once Alice receives her particle she performs a measurement on it. Alice is provided with two sets of measurement operators and she could choose to do one of the two measurements. These are labeled by P_{M1} and P_{M2} , respectively. Since Alice does not know in advance which measurement to apply, she adopts a random method to make her decision. Let us assume that each of these measurements can have two possible values $\{+1, -1\}$. Let $M1$ and $M2$ be the values revealed by the two measurements P_{M1} and P_{M2} . Similarly, Bob's measurements are labeled by P_{M3} and P_{M4} . Each of these $M1, M2, M3$ and $M4$ can have the values $\{+1, -1\}$. Bob does not decide in advance which measurement he will carry out and waits until he has received the particle from Alice and then chooses randomly. The setup is so arranged that they carry out their measurements in a causally connected manner. Thus the principle of no signalling ensures that the measurement of one particle cannot affect the measurement of the other.

Let us now consider the algebraic expression $(M1)(M3) + (M2)(M3) + (M2)(M4) - (M1)(M4)$. Since $M1, M2 = \pm 1$, it follows in either case that $(M1)(M3) + (M2)(M3) + (M2)(M4) - (M1)(M4) = \pm 2$. Now if consider the probability distribution and calculate the mean value of the above expression, then a little calculation gives the standard form of the Clauser-Horne-Shimony-Holt inequality [2]

$$E[(M1)(M3) + (M2)(M3) + (M2)(M4) - (M1)(M4)] = E[(M1)(M3)] + E[(M2)(M3)] + E[(M2)(M4)] - E[(M1)(M4)] \leq 2, \quad (1)$$

where E stands for the mean value. Interestingly, it can be seen that in standard quantum theory, it is always possible to design experiments for which this inequality gets violated [42–44]. This shows that quantum physics violates local realism, which implies that these results cannot be described by a LHV model. It may also provoke the implication that, if measurements on a quantum state violate a Bell's inequality, the state is entangled. However, with the advent of the Peres-Horodecki criteria [45, 46], the converse of this statement need not be true.

One can express the most general form of Bell-CHSH inequality for the mixed state $\rho = \frac{1}{4}[I \otimes I + (r \cdot \sigma) \otimes I + I \otimes (s \cdot \sigma) + \sum_{n,m=1}^3 t_{nm}(\sigma_n \otimes \sigma_m)]$ as $M(\rho) < 1$, where $M(\rho) = \max(u_i + u_j)$, u_i, u_j are the eigenvalues of the matrix $T^\dagger T$ (where the elements of the correlation matrix T is given by, $t_{mn} = \text{Tr}[\rho(\sigma_m \otimes \sigma_n)]$ and T^\dagger is the conjugate transpose of T) [19]. As mentioned above, violation of Bell's inequality for a given quantum state indicates that the state is entangled. But at the same time, there are certain entangled states which do not violate Bell's inequality.

Concurrence

Since Bell's inequality is not able to detect all possible entangled states, there is a need for some kind of measure which will quantify the amount of entanglement present in the system. A well known measure of entanglement is concurrence which for a two-qubit system, is equivalent to the entanglement of formation. The concurrence of a pure two-qubit state $|\psi_i\rangle$ is given by

$$C(|\psi_i\rangle) = \sqrt{2(1 - \text{Tr}\rho_A^2)} = \sqrt{2(1 - \text{Tr}\rho_B^2)}, \quad (2)$$

where $\rho_A = \text{Tr}_B|\psi\rangle\langle\psi|$ is the partial trace of $|\psi\rangle\langle\psi|$ over subsystem B, and ρ_B is the subsystem obtained when we trace out A. For a mixed state ρ , concurrence is defined as the optimization of average concurrence of all pure state decompositions of $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$,

$$C(\rho) = \min \sum_j p_j C(|\psi_j\rangle). \quad (3)$$

For a mixed state ρ of two qubits, concurrence is [4]

$$C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0), \quad (4)$$

where λ_i are the square root of the eigenvalues, in decreasing order, of the matrix $\rho^{\frac{1}{2}}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\rho^{\frac{1}{2}}$. ρ^* denotes complex conjugation of ρ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_y is the Pauli operator. The entanglement of formation (EOF) can then be expressed as a monotonic function of concurrence C as

$$E_F = -\frac{1 + \sqrt{1 - C^2}}{2} \log_2\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) - \frac{1 - \sqrt{1 - C^2}}{2} \log_2\left(\frac{1 - \sqrt{1 - C^2}}{2}\right). \quad (5)$$

Thus we see that concurrence, as a function of EOF, is a measure of quantum correlation as it actually quantifies the amount of entanglement present in the system.

Quantum Discord

As noted above, entanglement and quantum correlations need not be identical. Quantum discord attempts to measure and quantify all quantum correlations including entanglement [11–13].

In classical information theory, the correlation between two random variables X and Y is given by a quantity called ‘Mutual Information’ $J(X : Y) = H(X) - H(X|Y)$, where $H(X|Y)$ is the conditional entropy of X given that Y has already occurred and $H(X)$ is the Shannon entropy of the random variable X . Since $H(X|Y) = H(X, Y) - H(Y)$, there exists an alternative expression for mutual information $I(X : Y) = H(X) + H(Y) - H(X, Y)$. Classically, there is no ambiguity between these two expressions of mutual information and they are same. But a real difference arises in the quantum regime. In the quantum case, let us consider a bipartite state ρ_{XY} , where ρ_X and ρ_Y are the states of the individual subsystems. Here we note that in the quantum case, Shannon entropies $H(X)$, $H(Y)$ are replaced by von-Neumann entropies (e.g. $H(X) = H(\rho_X) = -\text{Tr}[\rho_X \log(\rho_X)]$). Now, from the definition itself the conditional entropy $H(X|Y)$ requires a specification of the state of X given the state of Y . Such a statement in quantum theory is ambiguous until the to-be-measured set of states of Y are selected. For that reason we focus on perfect measurements of Y defined by a set of one dimensional projectors $\{\pi_j^Y\}$. The subscript j is used for indexing different outcomes of this measurement. The state of X , after the measurement is given by

$$\rho_{X|\pi_j^Y} = \frac{\pi_j^Y \rho_{XY} \pi_j^Y}{\text{Tr}(\pi_j^Y \rho_{XY})}, \quad (6)$$

with probability $p_j = \text{Tr}(\pi_j^Y \rho_{XY})$. Thus, $H(\rho_{X|\pi_j^Y})$ is the von-Neumann entropy of the system in the state ρ_X , given that projective measurement is carried out on system Y in the most general basis $\{\cos(\theta)|0\rangle + \exp(i\phi)\sin(\theta)|1\rangle, \exp(-i\phi)\sin(\theta)|0\rangle - \cos(\theta)|1\rangle\}$. The entropies $H(\rho_{X|\pi_j^Y})$ weighted by the probabilities p_j , yield the

conditional entropy of X , given the complete set of measurements $\{\pi_j^Y\}$ on Y , as $H(X|\{\pi_j^Y\}) = \sum_j p_j H(\rho_{X|\pi_j^Y})$. From this, the quantum analogue of $J(X : Y)$ is seen to be

$$J(X : Y) = H(X) - H(X|\{\pi_j^Y\}), \quad (7)$$

while $I(X : Y)$ is similar to its classical counterpart

$$I(X : Y) = H(X) + H(Y) - H(X, Y). \quad (8)$$

It is clearly evident that these two expressions are not identical in quantum theory. Quantum discord is the difference between these two generalizations of classical mutual information,

$$D(X : Y) = H(Y) - H(X, Y) + H(X|\{\pi_j^Y\}). \quad (9)$$

The classical correlation is given by the difference between the mutual information and quantum correlation

$$C(X : Y) = I(X : Y) - D(X : Y). \quad (10)$$

We thus see from the above expression, that quantum discord aims to quantify the amount of quantum correlation that remains in the system and also points out that classicality and separability are not synonymous. In other words, it actually reveals the quantum advantage over the classical correlation.

Teleportation Fidelity

In addition to all these measures of quantum correlation one could also attempt to quantify them in terms of an application, for e.g., fidelity of teleportation [16]. It involves the separation of an input state into its classical and quantum part from which the state can be reconstructed with perfect fidelity $F = 1$. The basic idea is to use a pair of particles in a singlet state shared by sender (Alice) and receiver (Bob). Popescu [21] noticed that pairs in a mixed state could be still useful for (imperfect) teleportation. Consider the following general representation of the mixed state of a two-qubit system :

$$\rho = \frac{1}{4}[I \otimes I + (r \cdot \sigma) \otimes I + I \otimes (s \cdot \sigma) + \sum_{n,m=1}^3 t_{nm}(\sigma_n \otimes \sigma_m)], \quad (11)$$

where ρ acts on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and I stands for identity operator. Here $\mathcal{H}_1, \mathcal{H}_2$ are the individual Hilbert spaces of the two qubits and σ_n with $n = 1, 2, 3$ are the standard Pauli matrices, r, s are vectors in R^3 , $r \cdot \sigma = \sum_{i=1}^3 r_i \sigma_i$. The quantities $t_{nm} = \text{Tr}[\rho(\sigma_n \otimes \sigma_m)]$ are the coefficients of a real matrix denoted by T . This representation is most convenient when one talks about the inseparability of mixed states. In fact, all the parameters fall into two different classes: those that describe the local behavior of the state, i.e., (r and s), and those responsible for correlations (T matrix).

In the standard teleportation scheme we have a mixed state ρ acting as a quantum channel (originally formed by pure singlet states [16]). One of the particles is with Bob while the

other one and a third particle in an unknown state $|\phi\rangle$ are subjected to joint measurement in Alice's Hilbert space. These measurement operators are given by a family of projectors

$$P_k = |\psi_k\rangle\langle\psi_k|, k = 0, 1, 2, 3, \quad (12)$$

where ψ_k constitute the so-called Bell basis. Then using two bits Alice sends Bob the result of outcome k on basis of which he applies some unitary transformation U_k , obtaining in this way his particle in a state k . Then the fidelity of transmission of the unknown state is given by [19],

$$F = \int_S dR(\phi) \sum_k p_k \text{Tr}(\rho_k P_\phi), \quad (13)$$

where the integral is taken over all states (indexed by the angle ϕ) belonging to the Bloch sphere with uniform distribution R and $p_k = \text{Tr}[(P_k \otimes I)(P_\phi \otimes \rho)]$ denotes the probability of the k -th outcome. Now the task is to find those unitary transformations U_k that produce the highest fidelity (a choice of a quadruple of such U_k is what would be called a strategy). Maximizing F over all strategies gives [19]

$$\begin{aligned} F_{max} &= \frac{1}{2}(1 + \frac{1}{3}N(\rho)) \\ &= \frac{1}{2}(1 + \frac{1}{3}[\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3}]). \end{aligned} \quad (14)$$

Here u_i and u_j are the eigenvalues of $U = T^\dagger(\rho)T(\rho)$, where $T(\rho) = [T_{ij}]$, $T_{ij} = \text{Tr}[\rho(\sigma_i \otimes \sigma_j)]$ and T^\dagger implies the conjugate transpose of T . The classical fidelity of teleportation in the absence of entanglement is obtained as $\frac{2}{3}$. Thus whenever $F_{max} > \frac{2}{3}$ ($N(\rho) > 1$), teleportation is possible.

At this point it is quite interesting to note that there is a non-trivial interplay between Bell's inequality and teleportation fidelity. This is because both $M(\rho)$, $N(\rho)$ are dependent on the correlation matrix T . The relationship between these two quantities can be well understood in the form of the inequality $N(\rho) > M(\rho)$. Hence, it is clear that states which do violate Bell's inequality are always useful for teleportation. However, interestingly, this does not rule out the possibility of existence of entangled states that do not violate Bell's inequality, but can still be useful for teleportation. Indeed, we discuss below examples of such states.

A BRIEF INTRODUCTION TO OPEN QUANTUM SYSTEMS: QUANTUM NON DEMOLITION AND DISSIPATIVE TYPE

Open quantum systems is the study of evolution of the system of interest taking into account its interaction with the environment (reservoir or bath). Let the total Hamiltonian be $H = H_S + H_R + H_{SR}$, where S stands for the system, R for the reservoir and SR for the system-reservoir interaction. The evolution of the system of interest S is studied taking into account the effect of its environment R , through the SR interaction term, making the resulting dynamics non-unitary.

Based upon the system-reservoir interaction, open quantum systems can be broadly classified into two categories: (A). Quantum Non Demolition (QND), where $[H_S, H_{SR}] = 0$ resulting in decoherence without dissipation and (B). Dissipative, with $[H_S, H_{SR}] \neq 0$ resulting in decoherence along with dissipation [47]. Our model, for the study of quantum correlations, is a two-qubit system interacting with its environment, envisaged as a bath of non interacting harmonic oscillators, via QND or dissipative type of interactions. This provides the quantum channel used here, as in for e.g., teleportation, to study quantum correlations. The qubit-bath interactions are position dependent which leads to interesting collective behavior that can be broadly classified into independent (localized) or collective dynamical regimes. Below we discuss the model briefly, details of which can be found in [48, 49].

Two-Qubit QND Interaction with a Squeezed Thermal Bath

We consider the Hamiltonian, describing the QND interaction of two qubits with the bath as [47, 50, 51]

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= \sum_{n=1}^2 \hbar \varepsilon_n S_n^z + \sum_k \hbar \omega_k b_k^\dagger b_k \\ &\quad + \sum_{n,k} \hbar S_n^z (g_k^n b_k^\dagger + g_k^{n*} b_k). \end{aligned} \quad (15)$$

Here H_S , H_R and H_{SR} stand for the Hamiltonians of the system, reservoir and system-reservoir interaction, respectively. b_k^\dagger , b_k denote the creation and annihilation operators for the reservoir oscillator of frequency ω_k , S_n^z is the energy operator of the n th qubit, g_k^n stands for the coupling constant (assumed to be position dependent) for the interaction of the oscillator field with the qubit system and is taken to be

$$g_k^n = g_k e^{-ik \cdot r_n}, \quad (16)$$

where r_n is the qubit position. Since $[H_S, H_{SR}] = 0$, the Hamiltonian (1) is of QND type. In the parlance of quantum information theory, the noise generated is called the phase damping noise [47, 52].

The position dependence of the coupling of the qubits to the bath (16) helps to bring out the effect of entanglement between qubits through the qubit separation: $r_{mn} \equiv r_m - r_n$. This allows for a discussion of the dynamics in two regimes: (A). independent decoherence where $k \cdot r_{mn} \sim \frac{r_{mn}}{\lambda} \geq 1$ and (B). collective decoherence where $k \cdot r_{mn} \sim \frac{r_{mn}}{\lambda} \rightarrow 0$. The case (B) of collective decoherence would arise when the qubits are close enough for them to experience the same environment, or when the bath has a long correlation length (set by the effective wavelength λ) compared to the interqubit separation r_{mn} [50]. Our aim is to study the reduced dynamics of the qubit system. We assume separable initial conditions, i.e.,

$$\rho(0) = \rho^s(0) \otimes \rho_R(0), \quad (17)$$

where

$$\rho^s(0) = \rho_1^s(0) \otimes \rho_2^s(0), \quad (18)$$

is the initial state of the qubit system and the subscripts denote the individual qubits. In Eq. (17), $\rho_R(0)$ is the initial density matrix of the reservoir which we take to be a broadband squeezed thermal bath [47, 51, 52] given by

$$\rho_R(0) = S(r, \Phi) \rho_{th} S^\dagger(r, \Phi), \quad (19)$$

where

$$\rho_{th} = \prod_k [1 - e^{-\beta \hbar \omega_k}] e^{-\beta \hbar \omega_k b_k^\dagger b_k} \quad (20)$$

is the density matrix of the thermal bath at temperature T , with $\beta \equiv 1/(k_B T)$, k_B being the Boltzmann constant, and

$$S(r_k, \Phi_k) = \exp \left[r_k \left(\frac{b_k^2}{2} e^{-2i\Phi_k} - \frac{b_k^{\dagger 2}}{2} e^{2i\Phi_k} \right) \right] \quad (21)$$

is the squeezing operator with r_k, Φ_k being the squeezing parameters [53]. In order to obtain the reduced dynamics of the system, we trace over the reservoir variables [48]. The results pertaining to a thermal reservoir without squeezing can be obtained by setting the squeezing parameters to zero.

Two-Qubit Dissipative Interaction with a Squeezed Thermal Bath

We consider the Hamiltonian, describing the dissipative interaction of N qubits (two-level atomic system) with the bath (modeled as a 3-D electromagnetic field (EMF)) via the dipole interaction as [54]

$$\begin{aligned} H &= H_S + H_R + H_{SR} \\ &= \sum_{n=1}^N \hbar \omega_n S_n^z + \sum_{\vec{k}s} \hbar \omega_k (b_{\vec{k}s}^\dagger b_{\vec{k}s} + 1/2) \\ &\quad - i\hbar \sum_{\vec{k}s} \sum_{n=1}^N [\vec{\mu}_n \cdot \vec{g}_{\vec{k}s}(\vec{r}_n) (S_n^+ + S_n^-) b_{\vec{k}s} - h.c.]. \end{aligned} \quad (22)$$

Here $\vec{\mu}_n$ are the transition dipole moments, dependent on the different atomic positions \vec{r}_n and

$$S_n^+ = |e_n\rangle\langle g_n|, \quad S_n^- = |g_n\rangle\langle e_n|, \quad (23)$$

are the dipole raising and lowering operators satisfying the usual commutation relations and

$$S_n^z = \frac{1}{2}(|e_n\rangle\langle e_n| - |g_n\rangle\langle g_n|), \quad (24)$$

is the energy operator of the n th atom, while $b_{\vec{k}s}^\dagger, b_{\vec{k}s}$ are the creation and annihilation operators of the field mode (bath) $\vec{k}s$

with the wave vector \vec{k} , frequency ω_k and polarization index $s = 1, 2$ with the system-reservoir (S-R) coupling constant

$$\vec{g}_{\vec{k}s}(\vec{r}_n) = \left(\frac{\omega_k}{2\varepsilon\hbar V} \right)^{1/2} \vec{e}_{\vec{k}s} e^{i\vec{k} \cdot \vec{r}_n}. \quad (25)$$

Here V is the normalization volume and $\vec{e}_{\vec{k}s}$ is the unit polarization vector of the field. It can be seen from Eq. (25) that the S-R coupling constant is dependent on the atomic position r_n . This leads to a number of interesting dynamical aspects, as seen below. From now we will concentrate on the case of two qubits. Assuming separable initial conditions, and taking a trace over the bath, in a squeezed thermal state, the reduced density matrix of the qubit system in the interaction picture and in the usual Born-Markov, rotating wave approximation (RWA) is [54]

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{i}{\hbar} [H_S, \rho] - \frac{1}{2} \sum_{i,j=1}^2 \Gamma_{ij} [1 + \tilde{N}] \\ &\quad \times (\rho S_i^+ S_j^- + S_i^+ S_j^- \rho - 2S_j^- \rho S_i^+) \\ &\quad - \frac{1}{2} \sum_{i,j=1}^2 \Gamma_{ij} \tilde{N} (\rho S_i^- S_j^+ + S_i^- S_j^+ \rho - 2S_j^+ \rho S_i^-) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^2 \Gamma_{ij} \tilde{M} (\rho S_i^+ S_j^+ + S_i^+ S_j^+ \rho - 2S_j^+ \rho S_i^+) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^2 \Gamma_{ij} \tilde{M}^* (\rho S_i^- S_j^- + S_i^- S_j^- \rho - 2S_j^- \rho S_i^-). \end{aligned} \quad (26)$$

In Eq. (26)

$$\tilde{N} = N_{th} (\cosh^2(r) + \sinh^2(r)) + \sinh^2(r), \quad (27)$$

$$\tilde{M} = -\frac{1}{2} \sinh(2r) e^{i\Phi} (2N_{th} + 1) \equiv R e^{i\Phi(\omega_0)}, \quad (28)$$

with

$$\omega_0 = \frac{\omega_1 + \omega_2}{2}, \quad (29)$$

and

$$N_{th} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}. \quad (30)$$

Here N_{th} is the Planck distribution giving the number of thermal photons at the frequency ω and r, Φ are squeezing parameters. The analogous case of a thermal bath without squeezing can be obtained from the above expressions by setting these squeezing parameters to zero, while setting the temperature (T) to zero one recovers the case of the vacuum bath. Here the assumption of perfect matching of the squeezed modes to the modes of the EMF is made along with, the squeezing bandwidth being much larger than the atomic line widths. Also, the squeezing carrier frequency is taken to be tuned in resonance with the atomic frequencies.

In Eq. (26),

$$H_{\tilde{S}} = \hbar \sum_{n=1}^2 \omega_n S_n^z + \hbar \sum_{\substack{i,j \\ (i \neq j)}}^2 \Omega_{ij} S_i^+ S_j^-, \quad (31)$$

where

$$\Omega_{ij} = \frac{3}{4} \sqrt{\Gamma_i \Gamma_j} \left[-[1 - (\hat{\mu} \cdot \hat{r}_{ij})^2] \frac{\cos(k_0 r_{ij})}{k_0 r_{ij}} + [1 - 3(\hat{\mu} \cdot \hat{r}_{ij})^2] \left[\frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^2} + \frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^3} \right] \right]. \quad (32)$$

Here $\hat{\mu} = \hat{\mu}_1 = \hat{\mu}_2$ and \hat{r}_{ij} are unit vectors along the atomic transition dipole moments and $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, respectively. Also $k_0 = \omega_0/c$, with ω_0 being as in Eq. (29), $r_{ij} = |\vec{r}_{ij}|$. The wave vector $k_0 = 2\pi/\lambda_0$, λ_0 being the resonant wavelength, occurring in the term $k_0 r_{ij}$ sets up a length scale into the problem depending upon the ratio r_{ij}/λ_0 . This is thus the ratio between the inter atomic distance and the resonant wavelength, allowing for a discussion of the dynamics in two regimes: (A). independent decoherence where $k_0 \cdot r_{ij} \sim \frac{r_{ij}}{\lambda_0} \geq 1$ and (B). collective decoherence where $k_0 \cdot r_{ij} \sim \frac{r_{ij}}{\lambda_0} \rightarrow 0$. The case (B) of collective decoherence would arise when the qubits are close enough for them to feel the bath collectively or when the bath has a long correlation length (set by the resonant wavelength λ_0) in comparison to the interqubit separation r_{ij} . Ω_{ij} (32) is a collective coherent effect due to the multi-qubit interaction and is mediated via the bath through the terms

$$\Gamma_i = \frac{\omega_i^3 \mu_i^2}{3\pi \epsilon \hbar c^3}. \quad (33)$$

The term Γ_i is present even in the case of single-qubit dissipative system bath interaction [55, 56] and is the spontaneous emission rate, while

$$\Gamma_{ij} = \Gamma_{ji} = \sqrt{\Gamma_i \Gamma_j} F(k_0 r_{ij}), \quad (34)$$

where $i \neq j$ with

$$F(k_0 r_{ij}) = \frac{3}{2} \left[[1 - (\hat{\mu} \cdot \hat{r}_{ij})^2] \frac{\sin(k_0 r_{ij})}{k_0 r_{ij}} + [1 - 3(\hat{\mu} \cdot \hat{r}_{ij})^2] \times \left[\frac{\cos(k_0 r_{ij})}{(k_0 r_{ij})^2} - \frac{\sin(k_0 r_{ij})}{(k_0 r_{ij})^3} \right] \right]. \quad (35)$$

Γ_{ij} (34) is the collective incoherent effect due to the dissipative multi-qubit interaction with the bath. For the case of identical qubits, as considered here, $\Omega_{12} = \Omega_{21}$, $\Gamma_{12} = \Gamma_{21}$ and $\Gamma_1 = \Gamma_2 = \Gamma$.

QUANTUM CORRELATIONS IN OPEN QUANTUM SYSTEMS

Here we study quantum correlations, in two-qubit mixed states, generated by QND as well as dissipative system-reservoir interactions, as discussed above. We give examples

of entangled states that do not violate Bell's inequality, but can be useful for teleportation with a fidelity better than the classical fidelity of teleportation. On these states, we made a comparative study of various features of quantum correlation like teleportation fidelity (F_{max}), violation of Bell's inequality $M(\rho)$ (violation takes place for $M(\rho) \geq 1$), concurrence $C(\rho)$ and discord with respect to various experimental parameters like, bath squeezing parameter r , inter-qubit spacing r_{12} , temperature T and time of evolution t . Since these states are very close to Werner like states and since these Werner like states can be written in any basis, we measure the discord only in the computational basis $\{|0\rangle, |1\rangle\}$. Werner states being basis independent, the value of discord remains same in any basis. The motivation of this work is basically three fold. First of all, our motivation is to classify the set of two qubit states generated by a realistic open system models. We classify these density matrices into three broad categories: 1) Entangled states having positive concurrence, 2) Non classical separable states with a non zero discord and zero concurrence (entanglement) and 3) classical states i.e separable states with zero discord. In this process of classification of the generated states we also give a systematic overview of the threshold values of the dynamical parameters at which the transition of one category of states to the other category is taking place. Also, we look for realistic open system models that generate entangled states which can be useful for teleportation, but at the same time, not violate Bell's inequality. Finally in presence of dissipative decoherence we make a comparative study of the quantum and classical correlation. Interestingly we find that after a certain time the rate of quantum as well as classical decoherence becomes identical. An analogous study for non dissipative systems was made in [40].

Quantum Correlations in a QND System

Here we study examples of two-qubit density matrices obtained as a result of QND evolution, in both the collective as well as independent (localized) regimes of the model. Teleportation fidelity (F_{max}), violation of Bell's inequality $M(\rho)$, concurrence $C(\rho)$ and discord are examined for these states as a function of various parameters like: bath squeezing parameter r , inter-qubit spacing r_{12} , temperature T and system-bath interaction time t .

We categorize these density matrices into three broad categories: 1) entangled states, 2) separable non classical states and 3) classical states. We specifically mention the range of the dynamical parameters for which such a classification is possible. We also report cases where these two qubit states are entangled states, but do not violate Bell's inequality and still can be useful for teleportation with a fidelity better than the classical fidelity of teleportation.

The first example considers two-qubit density matrices, from the collective model, as illustrated in Figs. (1a), (b), (c) and (d), where the behavior of correlation measures: concurrence, teleportation fidelity, Bell's inequality violation and

discord, respectively, are studied with respect to the bath squeezing parameter r . As seen from Fig. (1(c)), $M(\rho) < 1$ for all values of the parameter, indicating that in the range shown, the states satisfy Bell's inequality and hence can be described by a local realistic model. Interestingly, for this range of r , Fig. (1(b)) shows that teleportation fidelity is greater than $\frac{2}{3}$, clearly indicating that the state is useful for teleportation. This is corroborated by Fig. (1(a)), where a plot of concurrence shows that concurrence is positive (indicating presence of entanglement), attaining a maximum value 0.13, approximately. Figure (1(d)), shows non-vanishing discord for the above range of the parameter r . Since both concurrence and discord attains non zero values for all range of the bath squeezing parameter r , it indicates that the two qubit density matrices obtained in this collective model is entangled for all values of r in this range. It can also be seen that, for the value of the various parameters chosen, with increase in bath squeezing, i.e., with increase in the absolute value of the parameter r , concurrence, teleportation fidelity, $M(\rho)$ decrease, but discord can increase.

The next example, depicted in Figs. (2(a)), (b), (c), (d), studies two-qubit density matrices, from the independent model. In Fig. (2(a)), concurrence is plotted with respect to bath squeezing parameter r . It is seen that states are entangled when r lies in the range $[-1.66, 1.66]$. Teleportation fidelity, as in the Fig. (2(b)), indicates that the states are useful for teleportation for the same range of r , i.e., when they are entangled. However, from Bell's inequality, as shown in Fig. (2(c)), in the same range, we see that the states do not violate Bell's inequality. Interestingly, from Fig. (2(d)), we find a non zero quantum discord in the range $[-3, 3]$ and particularly, in the range $[-3, -1.66] \cup [1.66, 3]$, i.e., where entanglement as depicted in Fig. (2(a)) is zero, discord is non-zero. Thus we conclude that the states obtained in this independent model are non classical separable states for the above range of the bath squeezing parameter r . This brings out the fact that the amount of entanglement present in a system is not equivalent to the total amount of quantum correlation in it. For the parameters chosen, we see a similar pattern of the various correlation functions as in the previous figure, i.e., with the exception of discord, the quantum correlation measures fall with the increase in bath squeezing.

Our last example from a QND evolution, in the independent regime of the model, is as shown in Figs. (3(a)), (b), (c), (d), where concurrence, teleportation fidelity, Bell's inequality and discord, respectively, are depicted with respect to the inter-qubit spacing r_{12} . In Fig. (3(a)), from the plot of concurrence, it can be seen that the states are entangled with a positive concurrence for all values of r_{12} , except in the range (1.4, 1.7), (4.6, 4.8). Like in the previous example, we see non-vanishing value of discord in the range where there is no entanglement, as shown in Fig. (3(d)). The states generated in this range are not entangled but they are non classical separable states.

Figure (3(b)) shows that the states useful for teleportation are from the same range of r_{12} , for which they are entangled.

From Fig. (3(c)), it can be seen that there are certain regions where Bell's inequality is violated $r_{12} \in [2.9, 3.1]$, but mostly it is satisfied.

Quantum Correlations in Dissipative Systems

Here we consider those two-qubit density matrices which are generated as a result of dissipative system-bath interaction. A comparative analysis of various features of quantum correlations, such as, concurrence $C(\rho)$, teleportation fidelity (F_{max}), violation of Bell's inequality $M(\rho)$ and discord, is made on these states, generated by the above density matrices, as a function of various parameters: bath squeezing parameter r , inter-qubit spacing r_{12} , temperature T and time of system-bath evolution t .

We classify the density matrices generated as a result of dissipative system-bath interaction into three categories: 1) entangled states, 2) separable non classical states and 3) classical states. Quite similar to the analysis of the non dissipative systems here also we provide the threshold values of the dynamical parameters for which the transition from one particular class to another particular class is taking place. We also find examples of states which are entangled but do not violate Bell's inequality and still can be useful for teleportation with a fidelity better than the classical fidelity of teleportation. Finally we study the overall dynamics of quantum and classical correlations in presence of dissipative decoherence.

For our first example, we study the behavior of correlation measures: concurrence, teleportation fidelity, Bell's inequality violation and discord, as illustrated in Figs. (4a), (b), (c) and (d), respectively, as a function of the inter-qubit distance r_{12} . From Fig. (4(a)), we find that the two qubit density matrix is entangled with a positive concurrence for $r_{12} < 0.5$. Moreover, from Fig. (4(d)), a positive discord is seen for the complete range of r_{12} , even in the range where there is no entanglement ($r_{12} \geq 0.5$). Interestingly, we see that discord attains a constant value .2544 in this range. We conclude from this non zero value of discord even in the absence of entanglement that the states in this range are non classical separable states. Figure (4(b)) illustrates that $F_{max} > \frac{2}{3}$, for all values of r_{12} except where there is no entanglement. In the absence of entanglement the teleportation fidelity attains the constant value $\frac{2}{3}$. However, from Fig. (4(c)) we find that $M(\rho) < 1$ for all values of r_{12} and attains a constant value 0.478 in the absence of entanglement. This clearly demonstrates that these states can be useful for teleportation despite the fact that they satisfy Bell's inequality.

As a function of the inter-qubit distance, the various correlation measures exhibit oscillatory behavior, in the collective regime of the model, but flatten out subsequently to attain almost constant values in the independent regime of the model. This oscillatory behavior is due to the strong collective behavior exhibited by the dynamics due to the relatively close proximity of the qubits in the collective regime [48, 49].

Next, we study the behavior of quantum correlations, in

two-qubit density matrices in the collective regime of the model, with respect to the evolution time t . In Fig. (5(a)), concurrence is seen to exhibit damped oscillations. Here we see that the states are entangled initially until the time evolution parameter t reaches the value 0.371. Figure (5(b)), for teleportation fidelity F_{max} , also shows a damped behavior and can be seen, in general, to be greater than $\frac{2}{3}$, except for the range $t \geq 0.371$ where there is no entanglement (zero concurrence), where it attains a constant value $\frac{2}{3}$. From Fig. (5(c)), we find that the states satisfy Bell's inequality for all values t . Discord, as in Fig. (5(d)), is always positive, though its value is decreasing with time. We conclude from the figure that the states that are generated in the collective regime are initially entangled but after a certain value of the time evolution parameter $t = 0.371$, the states produced are no longer entangled but are non classical separable states. The threshold value of the discord at $t = 0.371$ is 0.183.

In Fig. (6), we have an example of mixed states, belonging to the collective regime of the dissipative two-qubit model, where concurrence is zero for all values of the parameter r . These states satisfy Bell's inequality, as is evident from (6(c)) (it takes a constant value .338), but teleportation fidelity, Fig. (6(b)), is equal to $2/3$, which is consistent with our understanding that states that are not entangled cannot be used for teleportation. However, these states have a discord (6(d)) of about 0.127. So basically these states are non classical separable states.

In Figs. (7 (a), (b)) we have examples of mixed states, belonging to the collective and independent regimes of the two-qubit model, respectively. We study explicitly the evolution of the quantum correlation (quantum discord), quantum mutual information, concurrence and classical correlation with the passage of time. Here we see that unlike non dissipative dynamics [40], no transition from classical to quantum dechorence takes place. On the contrary both classical and quantum correlations are lost due to the interaction with the environment. Interestingly we find that in the absence of entanglement from a certain time $t > \bar{t}$, the classical correlation and the quantum discord becomes identical. This clearly indicates that after a certain time the rate of quantum and classical decoherence becomes same.

DISCUSSION

Here we provide the result of our study in a nutshell. Our first objective was to classify the states. In the following table we give the classification of the states in the context of various dynamical parameters.

TABLE I:

Figures	Entangled States	Non Classical Separable States	Classical States
Fig 1	Yes $\forall r$	No	No
Fig 2	Yes $r \in [-1.66, 1.66]$	Yes $r \in [-3, -1.66] \cup [1.66, 3]$	Yes $r \in [-5, -3] \cup [3, 5]$
Fig 3	Yes $r_{12} \in [1, 1.4] \cup [1.7, 4.6] \cup [4.8, 5]$	Yes $r_{12} \in (1.4, 1.7) \cup (4.6, 4.8)$	No
Fig 4	Yes $r_{12} \in [0.1, 0.5]$	Yes $r_{12} \in [0.5, 2]$	No
Fig 5	Yes $t \in [0.1, 0.371]$	Yes $t \in [0.371, 1.1]$	No
Fig 6	No	Yes $\forall r$	No

Our next motivation was to provide examples of entangled states that do not violate Bell's inequality, but can still be useful for teleportation. We summarize our findings in the following table.

TABLE II:

Figures	Bells Inequality Violation $M(\rho) > 1$	$(F_{max} > \frac{2}{3})$
Fig 1	No	Yes
Fig 2	No	Yes $r \in [-1.66, 1.66]$ No $r \in [-5, -1.66] \cup [1.66, 5]$
Fig 3	Yes $r_{12} \in [2.9, 3.1]$ No $r_{12} \in [1, 2.9) \cup (3.1, 5]$	Yes $r_{12} \in [1, 1.4] \cup [1.7, 4.6] \cup [4.8, 5]$ No $r_{12} \in (1.4, 1.7) \cup (4.6, 4.8)$
Fig 4	No	Yes $r_{12} \in [0.1, 0.5]$ No $r_{12} \in [0.5, 2]$
Fig 5	No	Yes $t \in [0.1, 0.371]$ $t \in [0.371, 1.1]$
Fig 6	No	No

Finally we studied the dynamics of quantum as well as classical correlation for states generated by dissipative open system interactions. From Figs. (7 (a), (b)) we obtain certain features that are unlike the QND model, that is, there is no sudden transition between quantum and classical loss of correlations in a composite system. On the contrary the rate of loss of correlations are same in both the cases after certain time.

CONCLUSION

In this work, we have tried to understand the nature of quantum correlations in mixed states. We focus on two-qubit systems, and the mixed states we work with are generated by concrete Open System models, with system-bath interactions such that both pure dephasing (QND) as well as dissipative evolutions are obtained.

Our primal objective was to classify the states, obtained as a result of both pure dephasing (QND) as well as dissipative evolutions into three categories: 1) Entangled states, 2) Non Classical Separable states, 3) Classical states. We were able to make this classification in the context of various dynamical parameters for both QND and dissipative systems.

One of our objectives was to see whether one can generate states, from realistic models, which satisfy Bell's inequality, that is, can be described by a local realistic theory, yet can be used for a useful task employing entanglement, viz. teleportation. We were able to give a number of such examples, both for the QND as well as dissipative type of evolutions.

In addition our objective was to see the dynamics of the quantum as well as classical correlations in the presence of dissipative decoherence. Analyzing the dynamics, we conclude that there do exist certain class of states obtained as result of a dissipative evolution for which the rate of decay of classical and quantum correlations are identical after a certain time interval.

These models, as well as the general evolutions considered, can be envisaged in an experimental setup [26, 27, 54]. This puts our study of quantum correlations on a firm basis.

* Electronic address: indranil@iopb.res.in

† Electronic address: subhashish@cmi.ac.in

‡ Electronic address: nanasid@cmi.ac.in

- [1] J. S. Bell, *Physics* **1**, 195 (1964).
- [2] J. F. Clauser and A. Shimony, *Rep. Progr. Phys.* **41**, 1881 (1978).
- [3] R. F. Werner, *Phys. Rev. A* **40**, 4277 (1989).
- [4] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
- [5] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu and R. Schack, *Phys. Rev. Lett.* **83**, 1054 (1999).
- [6] D. Kenigsberg, T. Mor and G. Ratsaby, *Quantum Inform. Comput.* **6**, 606 (2006).
- [7] E. Biham, G. Brassard, D. Kenigsberg and T. Mor, *Theor. Comput. Sci.* **320**, 15 (2004).
- [8] D. A. Meyer, *Phys. Rev. Lett.* **85**, 2014 (2000).
- [9] E. Knill and R. Laflamme, *Phys. Rev. Lett.* **81**, 5672 (1998).
- [10] A. Datta, A. Shaji and C. M. Caves, *Phys. Rev. Lett.* **100**, 050502 (2008).
- [11] H. Ollivier and W. H. Zurek, *Phys. Rev. Lett.* **88**, 017901 (2002).
- [12] L. Henderson and V. Vedral, *J. Phys. A* **34**, 6899 (2001).
- [13] S. Luo, *Phys. Rev. A* **77**, 042303 (2008).
- [14] K. Modi, T. Paterek, W. Son, V. Vedral and M. Williamson, *Phys. Rev. Lett.* **104**, 080501 (2010).
- [15] S. Luo, *Phys. Rev. A* **77**, 022301 (2008).
- [16] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
- [17] A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
- [18] N. Gisin, *Phys. Lett. A* **210**, 157 (1996).
- [19] R. Horodecki, P. Horodecki and M. Horodecki, *Phys. Lett. A* **222**, 21 (1996).
- [20] S. Massar and S. Popescu, *Phys. Rev. Lett.* **74**, 1259 (1995).
- [21] S. Popescu, *Phys. Rev. Lett.* **72**, 797 (1994).
- [22] I. Chakrabarty, *Eur. Phys. J. D* **57**, 265 (2010).
- [23] S. Adhikari, S. Roy, B. Ghosh, A. S. Majumdar and N. Nayak, *QIC* **10**, 0398 (2010).
- [24] S. Adhikari, N. Ganguly, I. Chakrabarty and B. S. Choudhury, *J. Phys. A: Math. Theor.* **41**, 415302 (2008).
- [25] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press 2002).
- [26] C. J. Myatt, B. E. King, Q. A. Turchette, C. A. Sackett, *et al.*, *Nature* **403**, 269 (2000).
- [27] Q. A. Turchette, C. J. Myatt, B. E. King, C. A. Sackett, *et al.*, *Phys. Rev. A* **62**, 053807 (2000).
- [28] M. S. Sarandy, *Phys. Rev. A* **80**, 022108 (2009); R. Dillenschneider, *Phys. Rev. B* **78**, 224413 (2008).
- [29] J. Maziero, L. C. Celeri, R. M. Serra and V. Vedral, *Phys. Rev. A* **80**, 044102 (2009).
- [30] J. Maziero, T. Werlang, F. F. Fanchini, L. C. Celeri and R. M. Serra, *Phys. Rev. A* **81**, 022116 (2010).
- [31] F. F. Fanchini, L. K. Castelano and A. O. Caldeira, eprint arXiv:0912.1468.
- [32] T. Werlang, S. Souza, F. F. Fanchini and C. J. Villas Boas, *Phys. Rev. A* **80**, 024103 (2009).
- [33] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda and A. O. Caldeira, *Phys. Rev. A* **81**, 052107 (2010).
- [34] D. O. Soares-Pinto, L. C. Celeri, R. Auccaise, F. F. Fanchini, *et al.*, *Phys. Rev. A* **81**, 062118 (2010).
- [35] J.-S. Xu, X.-Y. Xu, C.-F. Li, C.-J. Zhang, *et al.*, *Nat. Commun.* **10**, 1 (2010).
- [36] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, *et al.*, eprint, arXiv:1006.4344.
- [37] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, and A. Acin, *Phys. Rev. A* **81**, 052318 (2010).
- [38] A. Qasimi, D. F. V. James, A Comparison of the Attempts of Quantum Discord and Quantum Entanglement to Capture Quantum Correlations, eprint, arXiv:1007.1814.
- [39] L. Mazzola, B. Bellomo, R. L. Franco, G. Compagno, *Phys. Rev. A* **81**, 052116 (2010), e-print, arXiv:1003.5153.
- [40] L. Mazzola, J. Piilo, and S. Maniscalco, *Phys. Rev. Lett.* **104**, 200401 (2010).
- [41] M.D. Lang and C.M. Caves, *Phys. Rev. Lett.* **105**, 150501 (2010).
- [42] A. Aspect, J. Dalibard and G. Roger, *Phys. Rev. Lett.* **49**, 1804 (1982).
- [43] A. Aspect, P. Grangier and G. Roger, *Phys. Rev. Lett.* **49**, 91 (1982).
- [44] A. Aspect, P. Grangier and G. Roger, *Phys. Rev. Lett.* **47**, 460 (1981).
- [45] A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).
- [46] M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Lett. A* **223**, 1 (1996).
- [47] S. Banerjee and R. Ghosh, *J. Phys. A: Math. Theo.* **40**, 13735 (2007); eprint quant-ph/0703054.
- [48] S. Banerjee, V. Ravishanker and R. Srikanth, *Eur. Phys. J. D* **121**, 587 (2010).
- [49] S. Banerjee, V. Ravishanker and R. Srikanth, *Ann. of Phys. (N. Y.)* **128**, 588 (2010).

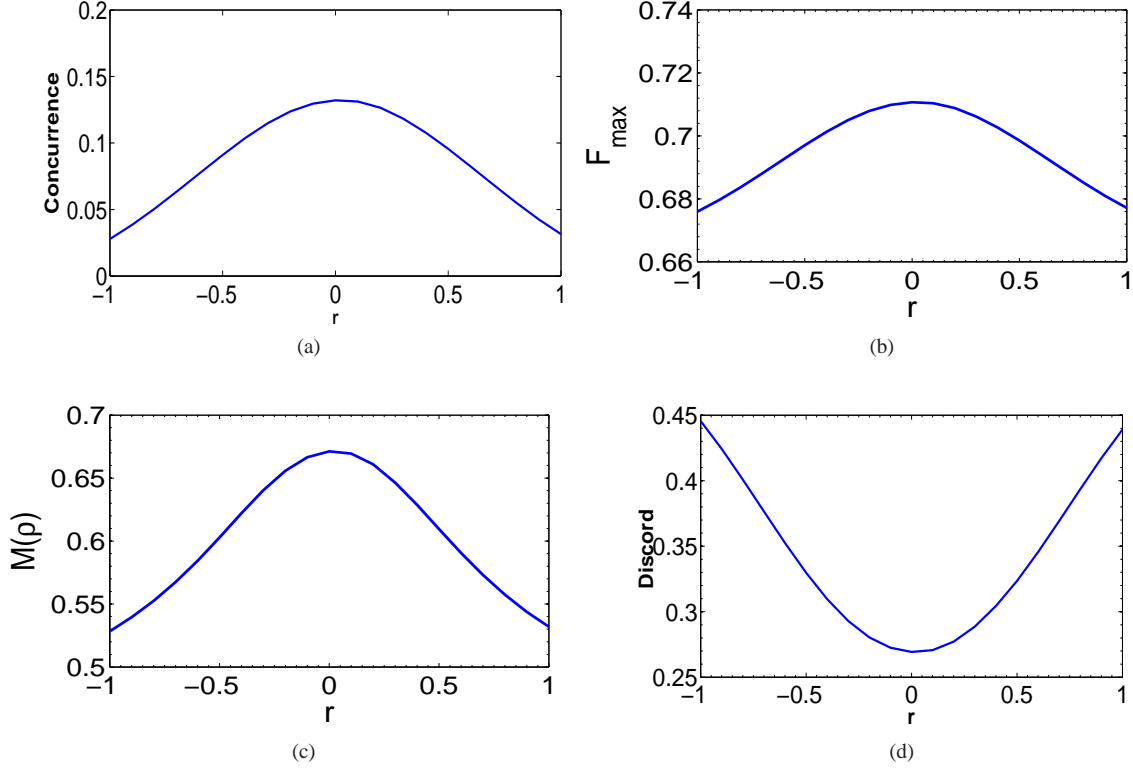


FIG. 1: Quantum correlations in a two-qubit system undergoing a QND interaction. The Figs. (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{\max} , test of Bell's inequality $M(\rho)$, discord as a function of bath squeezing parameter r . Here temperature T (in units where $\hbar \equiv k_B = 1$) is 25, evolution time t is 1 and the inter-qubit distance $r_{12} = 0.05$, i.e., the collective decoherence model.

- [50] J. H. Reina, L. Quiroga and N. F. Johnson, Phys. Rev. A **65**, 032326 (2002).
 [51] S. Banerjee, J. Ghosh and R. Ghosh, Phys. Rev. A **75**, 062106 (2007); eprint quant-ph/0703055.
 [52] S. Banerjee and R. Srikanth, Eur. Phys. J. D **46** 335 (2008); eprint quant-ph/0611161.
 [53] C. M. Caves and B. L. Schumacher, Phys. Rev. A **31**, 3068 (1985); B. L. Schumacher and C. M. Caves, Phys. Rev. A **31**,

3093 (1985).

- [54] Z. Ficek and R. Tanaś, Phys. Rep. **372**, 369 (2002).
 [55] S. Banerjee and R. Srikanth, Eur. Phys. J. D **121**, 587 (2009); eprint quant-ph/0611161.
 [56] S. Banerjee and R. Srikanth, Phys. Rev. A **76**, 062109 (2007); eprint: arXiv:0706.3633.

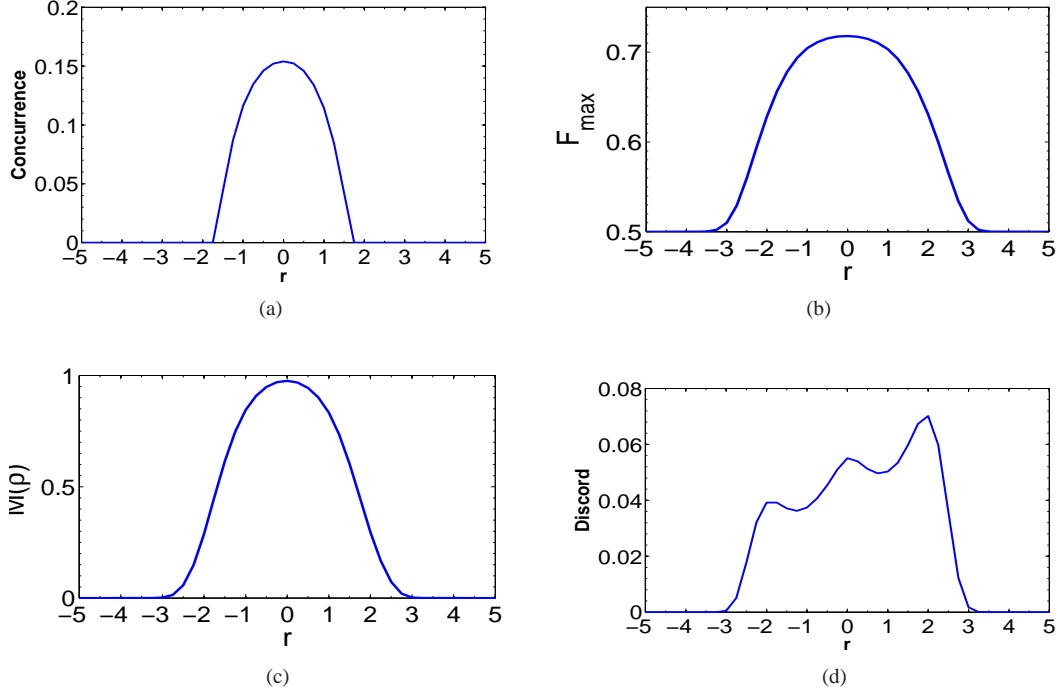


FIG. 2: Figures (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{\max} , test of Bell's inequality $M(\rho)$, and discord, respectively, as a function of bath squeezing parameter r , in a two-qubit system evolving via a QND interaction. Here temperature $T = 0$, evolution time t is 1.1 and the inter-qubit distance $r_{12} = 1.05$, i.e., the independent decoherence model.

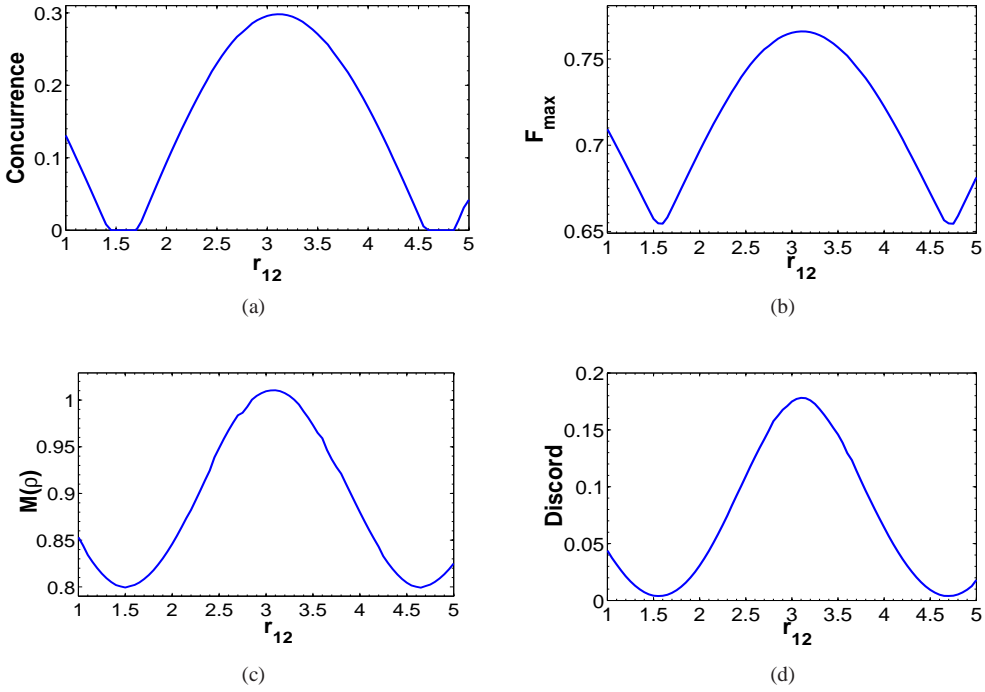


FIG. 3: Figures (a), (b), (c) and (d) represent the evolution, generated by a QND type of interaction, of concurrence, maximum teleportation fidelity F_{\max} , test of Bell's inequality $M(\rho)$, discord as a function of inter-qubit distance r_{12} . Here temperature $T = 0$, evolution time t is 1.1 and bath squeezing parameter $r = -1$.

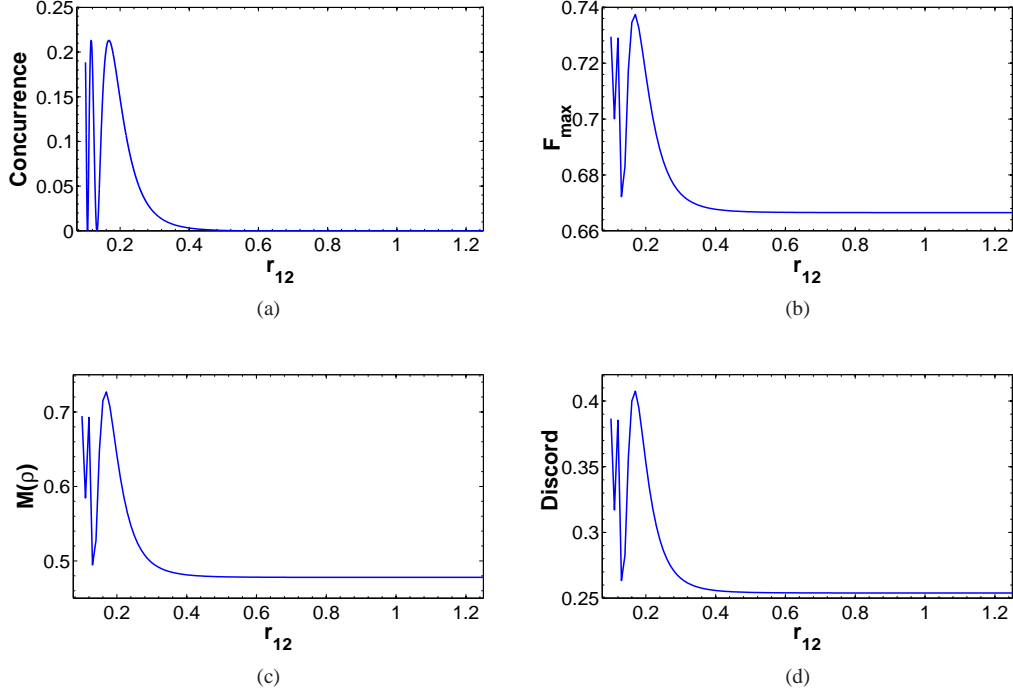


FIG. 4: Quantum correlations in a two-qubit system undergoing a dissipative evolution. The Figs. (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{\max} , test of Bell's inequality $M(\rho)$, discord as a function of inter-qubit distance r_{12} . Here temperature $T = 300$, evolution time t is 0.1 and bath squeezing parameter $r = -1$.

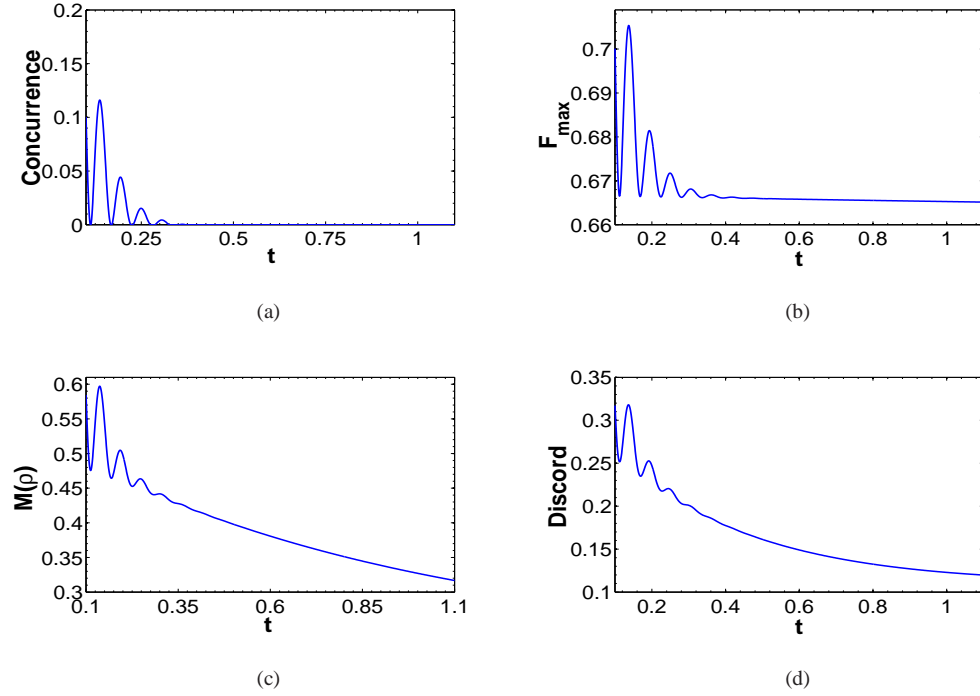


FIG. 5: Figures (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{\max} , test of Bell's inequality $M(\rho)$, discord with respect to the time of evolution t , evolving under a dissipative interaction. Here temperature $T = 300$, inter-qubit distance $r_{12} = 0.11$ and bath squeezing parameter $r = -1$.

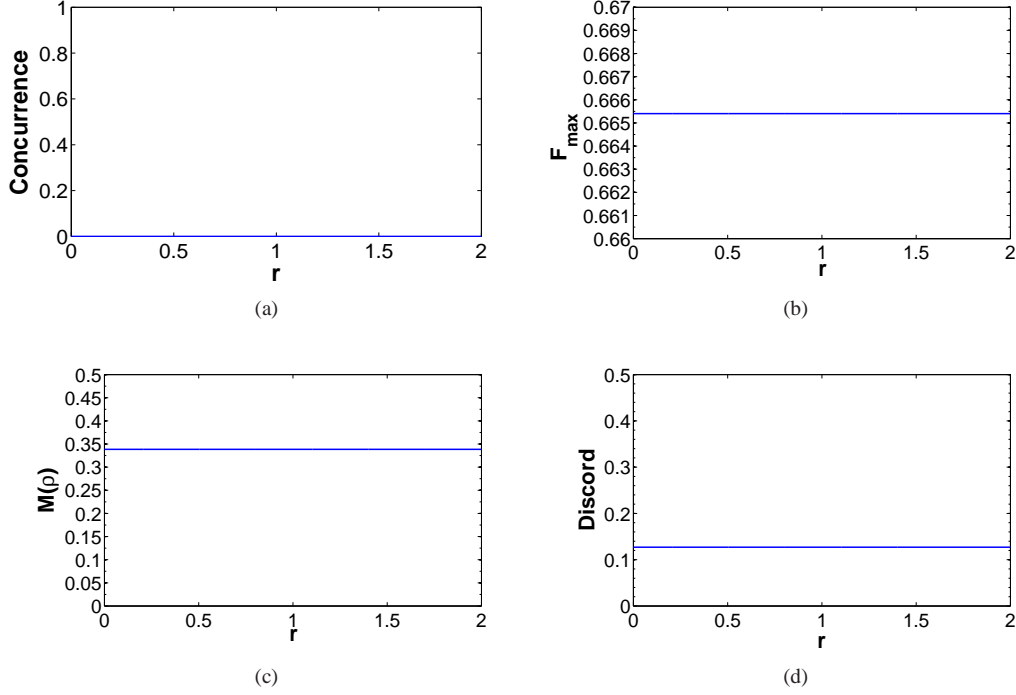


FIG. 6: An example showing vanishing entanglement, but non vanishing discord, for a dissipative two-qubit evolution. Figures (a), (b), (c) and (d) represent the evolution of concurrence, maximum teleportation fidelity F_{max} , test of Bell's inequality $M(\rho)$, discord with respect to the bath squeezing parameter r ranging from 0 to 2. Here temperature $T = 10$, $t = 0.9$ and inter-qubit distance $r_{12} = 0.09$.

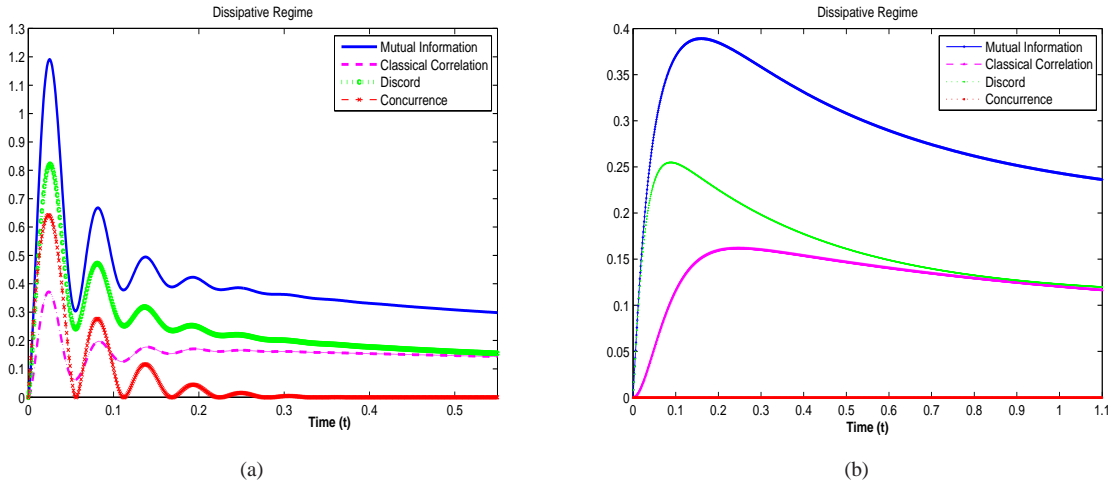


FIG. 7: An example showing vanishing entanglement, but non vanishing discord, for a dissipative two-qubit evolution. Figures (a) and (b) represent the evolution of mutual information (blue), quantum discord (quantum correlation) (green), concurrence (red) and classical information (pink) with respect to the time of evolution t , evolving under a dissipative interaction for collective and independent models, respectively. Here for (a) temperature $T = 10$, $r = 0$ and inter-qubit distance $r_{12} = 0.11$ and for (b) temperature $T = 10$, $r = 0$ and inter-qubit distance $r_{12} = 1.5$.